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# THE SELBERG ZETA FUNCTION FOR CONVEX CO-COMPACT SCHOTTKY GROUPS

LAURENT GUILLOPÉ, KEVIN K. LIN, AND MACIEJ ZWORSKI

**ABSTRACT.** We give a new upper bound on the Selberg zeta function for a convex co-compact Schottky group acting on  $\mathbb{H}^{n+1}$ : in strips parallel to the imaginary axis the zeta function is bounded by  $\exp(C|s|^\delta)$  where  $\delta$  is the dimension of the limit set of the group. This bound is more precise than the optimal global bound  $\exp(C|s|^{n+1})$ , and it gives new bounds on the number of resonances (scattering poles) of  $\Gamma \backslash \mathbb{H}^{n+1}$ . The proof of this result is based on the application of holomorphic  $L^2$ -techniques to the study of the determinants of the Ruelle transfer operators and on the quasi-self-similarity of limit sets. We also study this problem numerically and provide evidence that the bound may be optimal. Our motivation comes from molecular dynamics and we consider  $\Gamma \backslash \mathbb{H}^{n+1}$  as the simplest model of quantum chaotic scattering. The proof of this result is based on the application of holomorphic  $L^2$ -techniques to the study of the determinants of the Ruelle transfer operators and on the quasi-self-similarity of limit sets.

## 1. INTRODUCTION

In this paper we give an upper bound for the Selberg zeta function of a convex co-compact Schottky group in terms of the dimension of its limit set. This leads to a Weyl-type upper bound for the number of zeros of the zeta function in a strip with the number of degrees of freedom given by the dimension of the limit set plus one. We also report on numerical computations which indicate that our upper bound may be sharp, and close to a possible lower bound.

Our motivation comes from the study of the distribution of quantum resonances – see [36] for a general introduction. Since the work of Sjöstrand [31] on geometric upper bounds for the number of resonances, it has been expected that for chaotic scattering systems the density of resonances near the real axis can be approximately given by a power law with the power equal to half of the dimension of the trapped set (see (1.1) below). Upper bounds in geometric situations have been obtained in [33] and [35].

Recent numerical studies in the semi-classical and several convex obstacles settings, [11],[12] and [13] respectively, have provided evidence that the density of resonances satisfies a lower bound related to the dimension of the trapped set. In complicated situations which were studied numerically, the dimension is a delicate concept and it may be that different notions of dimension have to be used for upper and lower bounds – this point has been emphasized in [13].

Generally, the zeros of dynamical zeta functions are interpreted as the classical correlation spectrum [30]. In the case of *convex co-compact hyperbolic quotients*,  $X = \Gamma \backslash \mathbb{H}^{n+1}$  quantum resonances also coincide with the zeros of the zeta function – see [20]. The notion of the dimension of the trapped set is also clear as it is given by  $2(1 + \delta)$ . Here  $\delta = \dim \Lambda(\Gamma)$  is the dimension of the limit set of  $\Gamma$ , that is the set of accumulation points of any  $\Gamma$ -orbit in  $\mathbb{H}^{n+1}$ ,  $\Lambda(\Gamma) \subset \partial \mathbb{H}^{n+1}$ .

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Hence we may expect that

$$(1.1) \quad \sum_{|\operatorname{Im} s| \leq r, \operatorname{Re} s > -C} m_{\Gamma}(s) \sim r^{1+\delta},$$

where  $m_{\Gamma}(s)$  is the multiplicity of the zero of the zeta function of  $\Gamma$  at  $s$ .

Referring for definitions of Schottky groups and zeta functions to Sections 2 and 3 respectively we have

**Theorem.** *Suppose that  $\Gamma$  is a convex co-compact Schottky group and that  $Z_{\Gamma}(s)$  is its Selberg zeta function. Then for any  $C_0 > 0$  there exists  $C_1$  such that for  $|\operatorname{Re} s| < C_0$*

$$(1.2) \quad |Z_{\Gamma}(s)| \leq C_1 \exp(C_1 |s|^{\delta}), \quad \delta = \dim \Lambda(\Gamma).$$

The proof of this result is based on the *quasi-self-similarity* of limit sets of convex co-compact Schottky groups and on the application of holomorphic  $L^2$ -techniques to the study of the determinants of the Ruelle transfer operators.

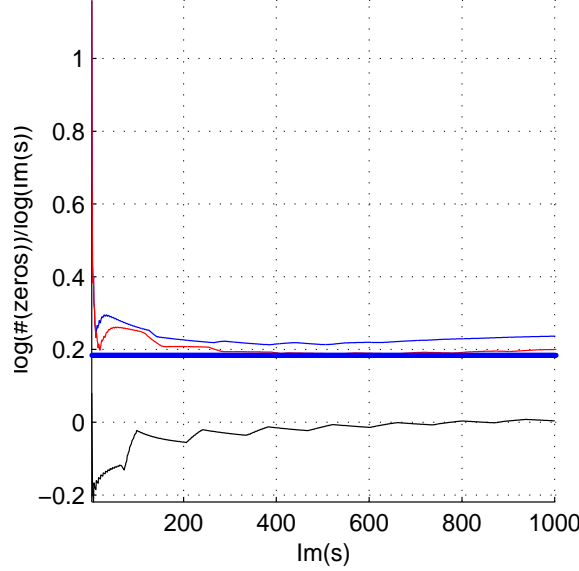


FIGURE 1. The plot of  $\log \log N(k) / \log k - 1$ , where  $N(k)$  is the number of zeros with  $|\operatorname{Im} s| \leq k$ , for a Schottky reflection group with  $\delta \simeq 0.184$ . Different lines represent different strips, and the thick blue line gives  $\delta$ .

If we use the convergence of the product representation (3.1) of the zeta function for  $\operatorname{Re} s$  large and apply Jensen's theorem we obtain the following

**Corollary 1.** *Let  $m_{\Gamma}(s)$  be the multiplicity of a zero of  $Z_{\Gamma}$  at  $s$ . Then, for any  $C_0$ , there exists some constant  $C_1$  such that for  $r > 1$*

$$(1.3) \quad \sum \{m_{\Gamma}(s) : r \leq |\operatorname{Im} s| \leq r+1, \operatorname{Re} s > -C_0\} \leq C_1 r^{\delta},$$

where  $\delta = \dim \Lambda(\Gamma)$ .

We can apply the preceding results to Schottky manifolds: a hyperbolic manifold is said Schottky if its fundamental group is Schottky. The case of surfaces being of special interest: any convex co-compact hyperbolic surface is Schottky. With the description of the divisor of the zeta function through spectral data established by Patterson and Perry [20], we can reformulate the preceding corollary nicely in the resonance setting. We do it only for surfaces (see below for short comments on higher dimensions)

**Corollary 2.** *Let  $M$  be a convex co-compact hyperbolic surface,  $\mathcal{S}_M$  be set of the scattering resonances of the Laplace-Beltrami operator on  $M$  and  $m_M(s)$  be the multiplicity of resonance  $s$ . Then, for any  $C_0$ , there exists some constant  $C_1$  such that for  $r > 1$*

$$(1.4) \quad \sum \{m_M(s) : s \in \mathcal{R}, r \leq |\operatorname{Im} s| \leq r+1, \operatorname{Re} s > -C_0\} \leq C_1 r^\delta,$$

where  $2(1+\delta)$  is the Hausdorff dimension of the recurrent set for the geodesic flow on  $T^*M$ .

This corollary is stronger than the result obtained in [35] where the upper bound of the type (1.1) was given. In fact, the upper bound (1.4) is what we would obtain had we had a Weyl law of the form  $r^{1+\delta}$  with a remainder  $\mathcal{O}(r^\delta)$ . That local upper bounds of this type are expected despite the absence of a Weyl law has been known since [23].

Section 7 deals with numerical computations of the density of zeros. They show that (1.1) may be true<sup>1</sup>. In fact, in the range of  $\operatorname{Im} s$  used in the computation we see that the number of zeros grows fast. If the range of  $\operatorname{Re} s$  is large (and fixed) we need very large  $\operatorname{Im} s$  to see the upper bound of Corollary 2. The computations also show that our bound on the zeta function is optimal. For values of  $Z_\Gamma(s)$  with  $\operatorname{Re} s$  negative we see that we need very large  $\operatorname{Im} s$  to see the onset of the upper bound. That is not surprising since we recall in Proposition 3.2 that  $\log |Z_\Gamma(s)| = \mathcal{O}(|s|^{n+1})$ , and that this bound is optimal (and of course  $\delta < n$ ).

We refer to Section 7 for the details and present here two pictures only. We take for  $\Gamma$  a group generated by compositions of reflections in three symmetrically spaced circles perpendicular to the unit circle, and cutting it at the angles  $30^\circ$  (see Fig.3 for the  $110^\circ$  angle). Fig.1 shows the density of zeros of  $Z_\Gamma$  in that case and Fig.2 plots the values of  $\log |\log |Z_\Gamma||$ .

## 2. SCHOTTKY GROUPS

The hyperbolic geometry on  $\mathbb{H}^{n+1}$  and the conformal geometry on its boundary at infinity  $\partial\mathbb{H}^{n+1} = \mathbb{S}^n$  share the same automorphism group: the isometry group  $\operatorname{Isom}(\mathbb{H}^{n+1})$  and the conformal group  $\operatorname{Conf}(\mathbb{S}^n)$  (with the conformal structure given by the standard metric on  $\mathbb{S}^n$  of curvature +1) are isomorphic. In particular any isometry  $g$  of  $\mathbb{H}^{n+1}$  induces on  $\mathbb{S}^n$  a conformal map  $\gamma$ , whose conformal distortion at the point  $w \in \mathbb{S}^n$  will be denoted by  $\|D\gamma(w)\|$ . There is also a correspondence between balls  $\mathcal{D}$  and spheres  $\mathcal{C}$  on  $\mathbb{S}^n$  (for  $n = 2$ , the original setting for Kleinian groups, these are discs and circles) and half-spaces  $\mathcal{P}$  and geodesic hyperplanes  $\mathcal{H}$  in  $\mathbb{H}^{n+1}$ :  $\mathcal{D} = \overline{\mathcal{P}} \cap \partial\mathbb{H}^{n+1}$  and  $\mathcal{C} = \overline{\mathcal{H}} \cap \partial\mathbb{H}^{n+1}$ . Given a hyperplane  $\mathcal{H}$  (a sphere  $\mathcal{C}$  resp.), its interior hyperplane (ball) will be given by the choice of a component of  $\mathbb{H}^{n+1} \setminus \mathcal{H}$  ( $\mathbb{S}^n \setminus \mathcal{C}$  resp.).

Let us review definitions of a Schottky group (see [17], [15], [28] and references given there), defined originally in conformal geometry on  $S^2$ . To the configuration of mutually disjoint geometric balls on the sphere  $S^n$ ,  $\mathcal{D}_i, i = 1, \dots, \ell$ , we associate the Schottky marked reflection group,

$$\Gamma(\mathcal{D}_1, \dots, \mathcal{D}_\ell),$$

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<sup>1</sup>Strictly speaking, the numerical computations are done for a slightly different zeta function of conformal dynamical system

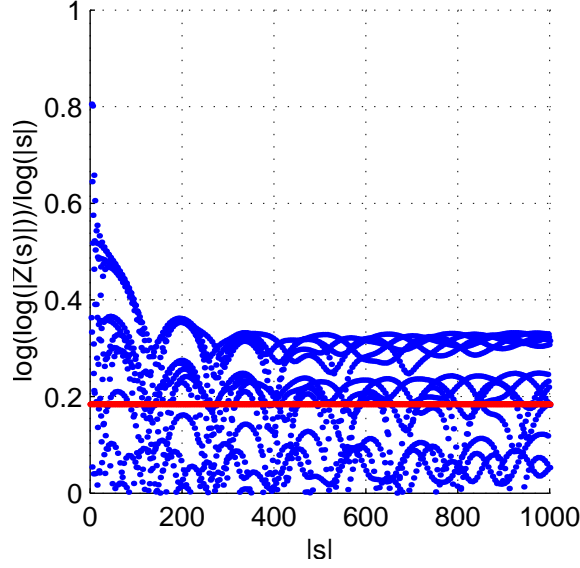


FIGURE 2. Density of values of  $\log |\log |Z_\Gamma|| / \log |s|$  for a Schottky reflection group with  $\delta \simeq 0.184$ .

defined as the group generated by the inversions  $\sigma_i$  in  $\partial\mathcal{D}_i, i = 1, \dots, \ell$ . The corresponding hyperbolic group is the Schottky marked reflection group

$$\Gamma(\mathcal{P}_1, \dots, \mathcal{P}_\ell),$$

for the collection,  $\mathcal{P}_1, \dots, \mathcal{P}_\ell$ , of mutually disjoint hyperbolic half-spaces  $\mathcal{P}_i$  with boundaries at infinity given by  $\mathcal{D}_i$ 's. The group is generated by the hyperbolic symmetries  $s_i$  in the hyperplane  $\partial\mathcal{P}_i$  (with infinite boundary  $\partial\mathcal{D}_i$ ),  $i = 1, \dots, \ell$ .

More generally, to a configuration of  $2\ell$  disjoint topological balls

$$\mathcal{D}_i, \quad i = 1, \dots, 2\ell,$$

with a set of conformal maps  $\gamma_i, i = 1, \dots, \ell$ , satisfying

$$\gamma_i(\mathbb{S}^n \setminus \mathcal{D}_i) = \overline{\mathcal{D}_{\ell+i}},$$

we associated the Schottky marked group

$$\Gamma(\mathcal{D}_1, \dots, \mathcal{D}_{2\ell}, \gamma_1, \dots, \gamma_\ell),$$

generated by the  $\gamma_i, i = 1, \dots, \ell$ .

We also introduce maps

$$\gamma_{\ell+j} = \gamma_j^{-1}, j = 1, \dots, \ell,$$

which map the exteriors of  $\mathcal{D}_{\ell+j}$ 's onto  $\overline{\mathcal{D}_j}$ .

The Schottky group  $\Gamma$  is a free group on the  $\ell$  (free) generators  $\gamma_1, \dots, \gamma_\ell$ . If the topological balls  $\mathcal{D}_i, i = 1, \dots, 2\ell$  are geometric balls, the marked Schottky group is said to be *classical*. A classical Schottky marked group  $\Gamma(\mathcal{D}_1, \dots, \mathcal{D}_{2\ell}, \gamma_1, \dots, \gamma_\ell)$ , subgroup  $\text{Conf}(\mathbb{S}^n)$ , can be presented as an isometry group of the hyperbolic space  $\mathbb{H}^{n+1}$ .

Let  $\mathcal{P}_i$  be the hyperbolic half-space with boundary at infinity given by  $\mathcal{D}_i$ . If  $g_i$  is the hyperbolic isometry of  $\mathbb{H}^{n+1}$  with action at infinity given by  $\gamma_i$ , then the classical Schottky group

$$\Gamma = \Gamma(\mathcal{D}_1, \dots, \mathcal{D}_{2\ell}, \gamma_1, \dots, \gamma_\ell),$$

has a hyperbolic marking

$$\Gamma = \Gamma(\mathcal{P}_1, \dots, \mathcal{P}_{2\ell}, g_1, \dots, g_\ell).$$

The Schottky domain  $\mathbb{H}^{n+1} \setminus \bigcup_{i=1}^{2\ell} \mathcal{P}_i$  is a fundamental domain for the action of  $\Gamma$  on  $\mathbb{H}^{n+1}$ .

A group is said to be a Schottky (reflection) group if it admits a presentation induced by a configuration of balls as described above. We remark that the subgroup of positive isometries of a Schottky reflection group is a classical Schottky group, with a marking

$$(\mathcal{D}_1, \dots, \mathcal{D}_{\ell-1}, \sigma_\ell \mathcal{D}_1, \dots, \sigma_\ell \mathcal{D}_{\ell-1}, g_1, \dots, g_{\ell-1}),$$

where  $g_i = \sigma_\ell \sigma_i$ . Such a group was called symmetrical by Poincaré [24].

An oriented hyperbolic manifold  $M$  is said to be (classical) Schottky if its fundamental group  $\pi_1(M)$  (realized as a discrete subgroup of  $\text{Isom}^+(\mathbb{H}^{n+1})$ ) admits a (classical) Schottky marking.

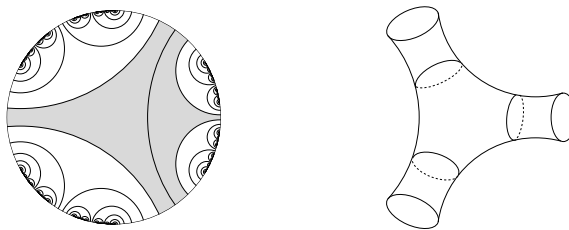


FIGURE 3. Tessellation by the group,  $\Gamma_\theta$ ,  $\theta = 110^\circ$ , generated by symmetries in three symmetrically placed lines each cutting the unit circle in an  $110^\circ$  angle, with the fundamental domain of its Schottky subgroup of direct isometries,  $\Gamma_\theta^+$ , and the associated Riemann surface  $\Gamma_\theta^+ \backslash \mathbb{H}^2$ . The dimension of the limit set is  $\delta = 0.70055063\dots$

A Schottky group is convex co-compact if in addition the closures  $\overline{\mathcal{D}_i}$  are mutually disjoint. In the convex co-compact case all non elliptic elements of  $\Gamma$  are hyperbolic. That means in particular that for  $\gamma \in \Gamma$  there exists  $\alpha \in \text{Isom}(\mathbb{H}^{n+1})$  such that, in the Poincaré model  $\mathbb{H}^{n+1} \simeq \mathbb{R}_+^{n+1} = \mathbb{R}_+ \times \mathbb{R}^n$ ,

$$(2.1) \quad \alpha^{-1} \gamma \alpha(x, y) = e^{\ell(\gamma)}(x, O(\gamma)y), \quad (x, y) \in \mathbb{R}_+^{n+1}, \quad O(\gamma) \in O(n), \quad \ell(\gamma) > 0.$$

If  $\Gamma \subset \text{Isom}^+(\mathbb{H}^{n+1})$ , the conjugacy classes of hyperbolic elements,

$$\{\gamma_1\} = \{\gamma_2\} \iff \exists \beta \in \Gamma \quad \beta \gamma_1 \beta^{-1} = \gamma_2,$$

are in one-to-one correspondence with closed geodesics of  $\Gamma \backslash \mathbb{H}^{n+1}$ . The primitive geodesics correspond to conjugacy classes of primitive elements of  $\Gamma$  (that is, elements which are not non-trivial powers). The magnification factor  $\exp(\ell(\gamma))$  in (2.1) gives the length  $\ell(\gamma)$  of the closed geodesic.

The limit set,  $\Lambda(\Gamma)$  of a discrete subgroup,  $\Gamma$ , of  $\text{Isom}(\mathbb{H}^{n+1})$ , is defined as the set in  $\overline{\mathbb{H}^{n+1}} = \mathbb{H}^{n+1} \cup \partial \mathbb{H}^{n+1}$  of accumulation points of any  $\Gamma$ -orbit in  $\mathbb{H}^{n+1}$ : the limit set  $\Lambda(\Gamma)$  is included in the boundary  $\partial \mathbb{H}^{n+1}$ . In the convex co-compact case it has a particularly nice structure; furthermore, for Schottky groups, it is totally disconnected and included in  $\mathcal{D} = \bigcup_{i=1}^{2\ell} \mathcal{D}_i$ . The aspects relevant to us come from the work of Patterson and Sullivan – see [32] and references

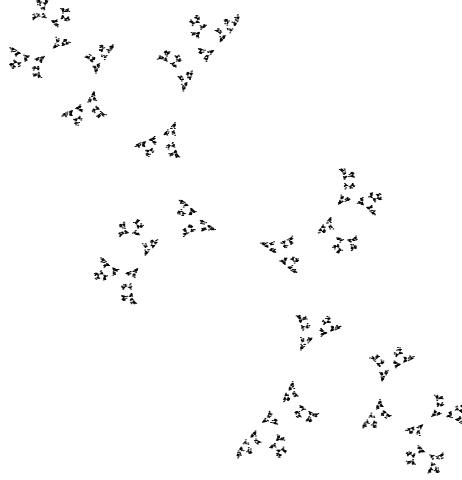


FIGURE 4. A typical limit set for a convex co-compact Schottky group,  $\Gamma \subset \text{Isom}(\mathbb{H}^3)$ .

given there. As will be discussed in more detail in Section 4, the limit set has a quasi-self-similar structure and a finite Hausdorff measure at dimension  $\delta = \delta(\Gamma)$ .

The limit set is related to the *trapped set*,  $K$ , of the usual scattering [31], [33], that is the set of points in phase space such that the trajectory through that point does not escape to infinity in either direction.: if  $\Delta$  is the diagonal of  $K \times K$  and  $\pi^*$  the projection from  $T^*\mathbb{H}^{n+1}$  on  $T^*\Gamma \backslash \mathbb{H}^{n+1}$ , the trapped set  $K$  is the union of the projections  $\pi^*(C_{\xi\eta})$  where  $C_{\xi\eta}$  is the geodesic with extremities  $\xi$  and  $\eta$ , both in the limit set  $\Lambda(\Gamma)$ . In particular, we have

$$\dim K = 2(\delta + 1),$$

see [35].

To stress the connection to closed orbits let us also mention that, generalizing earlier results of Guillopé [6] and Lalley [9], Perry [22] showed that

$$\#\{\{\gamma\} : \gamma \text{ primitive, } \ell(\gamma) < r\} \sim \frac{e^{\delta r}}{\delta r}.$$

### 3. PROPERTIES OF THE SELBERG ZETA FUNCTION

For  $\Gamma$ , a discrete subgroup of  $\text{Isom}(\mathbb{H}^{n+1})$ , the Selberg zeta function is defined as follows

$$(3.1) \quad Z_\Gamma(s) = \prod_{\{\gamma\}} \prod_{\alpha \in \mathbb{N}_0^n} \left( 1 - e^{-i\langle \theta(\gamma), \alpha \rangle} e^{-(s+|\alpha|)\ell(\gamma)} \right).$$

Here,  $\gamma \in \Gamma$  are hyperbolic,  $\exp(\ell(\gamma) + i\theta_j(\gamma))$  are the eigenvalues of the derivative of the action of  $\gamma$  on  $\mathbb{S}^n$  at the repelling fixed point ( $\exp(i\theta_j(\gamma))$  are the eigenvalues of the isometry  $O(\gamma)$  in the normal form (2.1)) and

$\{\gamma\}$  = the conjugacy class of a primitive hyperbolic element  $\gamma$ .

An element is called primitive if it is *not* a non-trivial power of another element.

We define the following map on  $\mathcal{D} = \bigcup_{i=1}^{2\ell} \mathcal{D}_i$ :

$$(3.2) \quad T : \mathcal{D} \longrightarrow \mathbb{S}^n, \quad T(x) = g_i(x), \quad x \in \mathcal{D}_i.$$

We need to find an open neighbourhood of the limit set where  $T$  is strictly expanding in the following sense :  $F$  defined on  $V$  is said to be (strictly) expanding on  $V$  with respect to the metric  $\| \cdot \|$  if there exists  $\theta \geq 1$  ( $\theta > 1$ ) such that

$$\|DF(v)\xi\| \geq \theta\|\xi\|, \quad v \in V, \xi \in T_v V.$$

In the case when  $\Gamma$  is the positive part of a Schottky reflection group, we can suppose that up to a conformal identification  $\partial\mathcal{D}_\ell$  is a great circle of the sphere  $\mathbb{S}^n$ . For the metric on  $\mathbb{S}^n$  we can take the metric induced by its embedding in  $\mathbb{R}^{n+1}$ . The inversion,  $\sigma_\ell$ , is the restriction to  $\mathbb{S}^n$  of the symmetry on  $\mathbb{R}^{n+1}$  with respect of the euclidean hyperplane countaining  $\partial\mathcal{D}_\ell$ , hence an isometry. Each inversion  $\sigma_i, i = 1, \dots, \ell - 1$  is expanding on the ball  $\mathcal{D}_i$ . Hence, the map  $T$  is expanding on  $V = \bigcup_{i=1}^{\ell-1} \mathcal{D}_i \cup \sigma_\ell \mathcal{D}_i$ , strictly expanding on any open set precompact in  $V$ .

However, the map  $T$  is not expanding on  $\mathcal{D}$  in general. To circumvent that we need to consider refinements,  $\mathcal{D}^N$ , defined by recurrence:

$$\begin{aligned} \mathcal{D}^1 &= \mathcal{D} \\ \mathcal{D}^N &= T^{-1}(\mathcal{D}^{N-1}) \cap \mathcal{D}^{N-1}. \end{aligned}$$

Each set  $\mathcal{D}^N$  is a disjoint union  $\bigcup_{i=1}^{d_N} \mathcal{D}_i^N$ . The collection of sets  $\{\mathcal{D}_i^N\}_i$  coincides with the collection

$$\{\mathcal{D}_\gamma\}_{\gamma \in \Gamma_N}, \quad \Gamma_N = \{\gamma \in \Gamma : |\gamma| = N\}.$$

Here  $|\gamma|$  is the combinatorial length with respect to the system of generators  $\{g_1, \dots, g_{2\ell}\}$ . In other words,

$$\mathcal{D}_\gamma = x_N^{-1} \dots x_2^{-1} \mathcal{D}_{x_1} \quad \text{if} \quad \gamma = x_1 \dots x_N,$$

where  $\mathcal{D}_{x_N} = \mathcal{D}_i$  if  $x_N = g_i$ . The iterated map,  $T^N$ , is defined on  $\mathcal{D}^N$ , and

$$T^{|\gamma|}|_{\mathcal{D}_\gamma} = \gamma,$$

The map  $T$  is strictly expanding on  $\mathcal{D}^N$  for  $N$  big enough as explained in the following lemma (see Lemma 9.2 in Lalley [9] for a similar result).

**Lemma 3.1.** *Let  $\Gamma$  a Schottky group and  $\mathcal{D}, T$  defined as in (3.2). There exist an integer  $N \geq 1$ , a metric  $\| \cdot \|_\Gamma$ , defined on  $\mathcal{D}^N$ , and a real  $\beta > 1$  such that  $\|DT(w)\|_\Gamma \geq \beta, w \in \mathcal{D}^N$ . The metric can be taken analytic on  $\mathcal{D}^N$ .*

*Proof.* Let us recall that any Möbius transformation  $\gamma$  of  $\mathbb{R}^k$  which does not fix the point at infinity,  $\infty$ , has an isometric sphere  $S_\gamma$ , and that  $\gamma^{-1}$  is strictly contracting on any compact subset of its exterior (the unbounded component of  $\mathbb{R}^k \setminus S_\gamma$ ). The sphere  $S_\gamma$  is centered at  $\gamma^{-1}\infty$  and if  $r_\gamma$  is its radius, we have (see [28])

$$(3.3) \quad \|\gamma x - \gamma y\| = \frac{r_\gamma^2 \|x - y\|}{\|x - \gamma^{-1}\infty\| \|y - \gamma^{-1}\infty\|}, \quad x, y \in \mathbb{R}^k \setminus \{\gamma^{-1}\infty\}$$

Up to a conformal transformation, we can suppose that  $\Gamma$  is a subset of  $\text{Conf}(\mathbb{R}^n)$ , with the point at infinity in its ordinary set. No non trivial element in  $\Gamma$  fixes  $\infty$  and, taking in (3.3) as  $x, y$  points in the upper half-plane  $\mathbb{H}^{n+1}$  (and the Poincaré extension of  $\gamma$  to  $\mathbb{R}^{n+1}$ ), we deduce that the set of radii  $\{r_\gamma, \gamma \in \Gamma\}$  accumulates only at 0.

For  $\gamma = x_1 \dots x_N$ , we have

$$\gamma^{-1}(\infty) \in \mathcal{D}_\gamma \subset \mathcal{D}_{x_2 \dots x_N} \subset \dots \subset \overline{\mathcal{D}_{x_{N-1} x_N}} \subset \mathcal{D}_{x_N}$$



hence there exists  $N_0$  such that the isometric sphere  $S_\gamma$  is included in  $\mathcal{D}_{x_N}$  if  $|\gamma| \geq N_0$ . For such a  $\gamma$ , the interior of the isometric sphere  $S_{\gamma^{-1}}$  is included in  $\mathcal{D}_{x_1^{-1}}$ : its exterior contains all the  $\overline{\mathcal{D}_{x_i}}, x_i \neq x_1^{-1}$ , hence  $\gamma$  is strictly contracting on  $\cup_{x_i \neq x_1^{-1}} \mathcal{D}_{x_i}$ . As  $\gamma(\mathcal{D}_\gamma) \subset \cup_{x_i \neq x_1^{-1}} \mathcal{D}_{x_i}$ , the map  $\gamma = T^{|\gamma|}|_{\mathcal{D}_\gamma}$  is expanding on  $\mathcal{D}_\gamma \subset \mathcal{D}^{|\gamma|}$ . We have just proved the existence of  $\eta_0 > 1$  such that  $\|DT^{N_0}(w)\| \geq \eta_0, w \in \mathcal{D}^{N_0}$ , hence there exist constants  $C > 0, \theta > 1$  such that  $\|DT^p(w)\| \geq C\theta^p, w \in \mathcal{D}^p, p \geq 1$ .

Taking an integer  $N$  such that  $C\theta^N > 1$ , we define on  $\mathcal{D}^N$  the metric (introduced by Mather [16])

$$\|V\|_\Gamma = \sum_{p=0}^{N-1} \|DT^p(w)V\|, \quad V \in T_w \mathcal{D}^N,$$

which concludes the proof.  $\square$

Let  $\tilde{\mathbb{S}}^n$  be a Grauert tube of  $\mathbb{S}^n$ , that is a complex  $n$ -manifold containing  $\mathbb{S}^n$  as a totally real submanifold (that is all we need). Let us then choose open neighbourhoods<sup>2</sup>  $D_i, i = 1, \dots, d_N$  of  $\mathcal{D}_i^N$  in  $\tilde{\mathbb{S}}^n$ . By further shrinking, we can suppose that the open sets  $D_i$  are mutually disjoint, and that the real analytic maps  $T$  and  $\|DT\|_\Gamma$  extend holomorphically to  $D = \cup_{i=1}^{d_N} D_i$ , with  $\|DT\|_\Gamma \geq \tilde{\beta}$  for some  $\tilde{\beta} > 1$ . The open sets  $D_i$  can be chosen to be a union  $D_i = \cup_{k=1}^{\delta_i} D_{ik}$  of open sets, each one biholomorphic to the ball  $B_{\mathbb{C}^n}(0, 1)$  in  $\mathbb{C}^n$ .

With this formalism in place we define the Ruelle transfer operator

$$(3.4) \quad \begin{aligned} \mathcal{L}(s)u(z) &= \sum_{Tw=z} \|DT(w)\|^{-s} u(w), \quad z \in D, \\ u &\in H^2(D), \quad H^2(D) = \{u \text{ holomorphic in } D : \iint_D |u(z)|^2 dm(z) < \infty\}. \end{aligned}$$

The only difference from the standard definition lies in choosing  $L^2$  spaces of holomorphic functions instead of Banach spaces. However we still obtain the analogue of a (special case of a) result of Ruelle [29] and Fried [4]:

**Proposition 3.2.** *Suppose that  $\mathcal{L}(s) : H^2(D) \rightarrow H^2(D)$  is defined by (3.4). Then for all  $s \in \mathbb{C}$   $\mathcal{L}(s)$  is a trace class operator and*

$$(3.5) \quad |\det(I - \mathcal{L}(s))| \leq \exp(C|s|^{n+1}).$$

*Proof.* The proof is based on estimates of the characteristic values,  $\mu_\ell(\mathcal{L}(s))$ . We will show that there exists  $C > 0$  such that

$$(3.6) \quad \mu_\ell(\mathcal{L}(s)) \leq C e^{C|s| - \ell^{\frac{1}{n}}/C}.$$

To see how that is obtained and how it implies (3.5) let us first recall some basic properties of characteristic values of a compact operator  $A : H_1 \rightarrow H_2$  where  $H_j$ 's are Hilbert spaces. We define

$$\|A\| = \mu_0(A) \geq \mu_1(A) \geq \dots \geq \mu_\ell(A) \rightarrow 0,$$

to be the eigenvalues of  $(A^*A)^{\frac{1}{2}} : H_1 \rightarrow H_1$ , or equivalently of  $(AA^*)^{\frac{1}{2}} : H_2 \rightarrow H_2$ . The min-max principle shows that

$$(3.7) \quad \mu_\ell(A) = \min_{\substack{V \subset H_1 \\ \text{codim } V = \ell}} \max_{\substack{v \in V \\ \|v\|_{H_1} = 1}} \|Av\|_{H_2}.$$

---

<sup>2</sup>we drop the index  $N$  in the open sets  $D_i^N$  for the purpose of notational simplicity

The following rough estimate will be enough for us here: suppose that  $\{x_j\}_{j=0}^\infty$  is an orthonormal basis of  $H_1$ , then

$$(3.8) \quad \mu_\ell(A) \leq \sum_{j=\ell}^\infty \|Ax_j\|_{H_2}.$$

To see this we will use  $V_\ell = \text{span} \{x_j\}_{j=\ell}^\infty$  in (3.7): for  $v \in V_\ell$  we have, by the Cauchy-Schwartz inequality, and the obvious  $\ell^2 \subset \ell^1$  inequality,

$$\|Av\|_{H_2}^2 = \left\| \sum_{j=\ell}^\infty \langle v, x_j \rangle_{H_1} Ax_j \right\| \leq \|v\|_{H_1}^2 \left( \sum_{j=\ell}^\infty \|Ax_j\|_{H_2} \right)^2,$$

from which (3.7) gives (3.8).

We will also need some real results about characteristic values. The first is the *Weyl inequality* (see [5], and also [31, Appendix A]). It says that if  $H_1 = H_2$  and  $\lambda_j(A)$  are the eigenvalues of  $A$ ,

$$|\lambda_0(A)| \geq |\lambda_1(A)| \geq \cdots \geq |\lambda_\ell(A)| \rightarrow 0,$$

then for any  $N$ ,

$$\prod_{\ell=0}^N (1 + |\lambda_\ell(A)|) \leq \prod_{\ell=0}^N (1 + |\mu_\ell(A)|).$$

In particular if the operator  $A$  is of trace class, that is if,  $\sum_\ell \mu_\ell(A) < \infty$ , then the determinant

$$\det(I + A) \stackrel{\text{def}}{=} \prod_{\ell=0}^\infty (1 + \lambda_\ell(A)),$$

is well defined and

$$(3.9) \quad |\det(I + A)| \leq \prod_{\ell=0}^\infty (1 + \mu_\ell(A)).$$

We also need to recall the following standard inequality about characteristic values (see [5]):

$$(3.10) \quad \mu_{\ell_1+\ell_2}(A+B) = \mu_{\ell_1}(A) + \mu_{\ell_2}(B)$$

We finish the review, as we started, with an obvious equality: suppose that  $A_j : H_{1j} \rightarrow H_{2j}$  and we form  $\bigoplus_{j=1}^J A_j : \bigoplus_{j=1}^J H_{1j} \rightarrow \bigoplus_{j=1}^J H_{2j}$ , as usual,  $\bigoplus_{j=1}^J A_j(v_1 \oplus \cdots \oplus v_J) = A_1 v_1 \oplus \cdots \oplus A_J v_J$ . Then

$$(3.11) \quad \sum_{\ell=0}^\infty \mu_\ell \left( \bigoplus_{j=1}^J A_j \right) = \sum_{j=1}^J \sum_{\ell=0}^\infty \mu_\ell(A_j).$$

With these preliminary facts taken care of, we see that (3.6) implies (3.5). In fact, (3.9) shows that

$$|\det(I - \mathcal{L}(s))| \leq \prod_{\ell=0}^\infty (1 + e^{C|s| - \ell^{\frac{1}{n}}/C}) \leq e^{C_1|s|^{n+1}}.$$

Hence it remains to establish (3.6). For that we will write

$$H^2(D) = \bigoplus_{i=1}^{d_N} H^2(D_i),$$

and introduce, for  $i, j = 1, \dots, d^N$ , the operator

$$\mathcal{L}_{ij}(s) : H^2(D_i) \rightarrow H^2(D_j),$$

non zero only when  $T(D_i)$  and  $D_j$  are not disjoint, where

$$(3.12) \quad \mathcal{L}_{ij}(s)u(z) \stackrel{\text{def}}{=} \|Df_{ij}(z)\|^s u(f_{ij}(z)), \quad z \in D_j, \quad f_{ij} = (T|_{D_i})^{-1}|_{D_j}.$$

From (3.10) and a version of (3.11) we then have

$$\mu_\ell(\mathcal{L}(s)) \leq \max_{1 \leq i, j \leq d^N} 2\mu_{[\ell/C]}(\mathcal{L}_{ij}(s)).$$

To estimate  $\mu_k(\mathcal{L}_{ij}(s))$ , let us recall that  $D_i$  was taken as an union of open sets  $D_{ik}, k = 1, \dots, \delta_i$  biholomorphic to  $B_{\mathbb{C}^n}(0, 1)$ : as  $f_{ij}(D_j)$  is relatively compact in  $D_i$ , we can find  $\rho \in (0, 1)$  (independent of  $i, j = 1, \dots, d^N$ ) such that  $f_{ij}(D_j) \subset D_i^\rho$  where  $D_i^\rho = \cup_{k=1}^{\delta_i} D_{ik}^\rho$  with  $D_{ik}^\rho \subset D_{ik}$  the pullback of the ball  $B_{\mathbb{C}^n}(0, \rho)$  through the biholomorphism of  $D_{ik}$  onto  $B_{\mathbb{C}^n}(0, 1)$ . The map  $\mathcal{L}_{ij}(s)$  is the composition

$$H^2(D_i) \xrightarrow{R} \oplus_{k=1}^{\delta_i} H^2(D_{ik}) \xrightarrow{\oplus R_{ik}^\rho} \oplus_{k=1}^{\delta_i} H^2(D_{ik}^\rho) \xrightarrow{\pi} H^2(D_i^\rho) \xrightarrow{\mathcal{L}_{ij}^\rho(s)} H^2(D_j)$$

where  $R$  and  $R_{ik}^\rho$  are the natural restrictions,  $\pi^\rho$  is the orthogonal projection on the space  $H^2(D_i^\rho)$  immersed in  $\oplus_{k=1}^{\delta_i} H^2(D_{ik}^\rho)$  by the natural restrictions and  $\mathcal{L}_{ij}^\rho(s)$  is defined by the same formula (3.12) as  $\mathcal{L}_{ij}(s)$ . The maps  $R$  and  $\pi^\rho$  are bounded, while the norm of  $\mathcal{L}_{ij}(s)$  is bounded by  $Ce^{C|s|}$ . The bounds on the singular values of  $R_{ik}^\rho$ , given up to a bounded factor by the following lemma, give the bound

$$\mu_\ell(\mathcal{L}_{ij}(s)) \leq Ce^{C|s| - \ell^{1/n}/C},$$

for some  $C$ , which completes the proof of (3.6).  $\square$

**Lemma 3.3.** *Let  $\rho \in (0, 1)$  and  $R_\rho : H^2(B_{\mathbb{C}^n}(0, 1)) \rightarrow H^2(B_{\mathbb{C}^n}(0, \rho))$  induced by the restriction map of  $B_{\mathbb{C}^n}(0, 1)$  to  $B_{\mathbb{C}^n}(0, \rho)$ . Then, for any  $\tilde{\rho} \in (\rho, 1)$  there exists a constant  $C$  such that*

$$\mu_\ell(R^\rho) \leq C\tilde{\rho}^{\ell^{1/n}}.$$

*Proof.* We use (3.8) with the standard basis  $(x_\alpha)_{\alpha \in \mathbb{N}^n}$  of  $H^2(B_{\mathbb{C}^n}(0, 1))$ :

$$(3.13) \quad x_\alpha(z) = c_\alpha z_1^{\alpha_1} \cdots z_n^{\alpha_n}, \quad \int_{B_{\mathbb{C}^n}(0, 1)} |x_\alpha(z)|^2 dm(z) = 1, \quad \alpha \in \mathbb{N}_0^n,$$

for which we have

$$\|R_\rho(x_\alpha)\|^2 = \int_{B_{\mathbb{C}^n}(0, \rho)} |x_\alpha(w)|^2 dm(w) = \rho^{2(|\alpha|+n)}.$$

The number of  $\alpha$ 's with  $|\alpha| \leq m$  is approximately  $m^n$  and hence by (3.8) we have

$$\mu_\ell(R_\rho) \leq C \sum_{Ck \geq \ell^{1/n}} k^{n-1} \rho^k \leq C\tilde{\rho}^{\ell^{1/n}}.$$

$\square$

The next proposition is a modification of standard zeta function arguments – see [25] and [26] for the discussion of the hyperbolic case.

**Proposition 3.4.** *Let  $\mathcal{L}(s)$  be defined by (3.4). Then, if  $Z_\Gamma$  is the zeta function (3.1) corresponding to the group  $\Gamma$  then*

$$Z_\Gamma(s) = \det(I - \mathcal{L}(s)).$$

*Proof.* For  $s$  fixed and  $z \in \mathbb{C}$

$$h(z) \stackrel{\text{def}}{=} \det(I - z\mathcal{L}(s))$$

is, in view of (3.6) and (3.9), an entire function of order 0. For  $|z|$  sufficiently small  $\log(I - z\mathcal{L}(s))$  is well defined and we have

$$(3.14) \quad \det(I - z\mathcal{L}(s)) = \exp \left( - \sum_{n=1}^{\infty} \frac{z^n}{n} \text{tr}(\mathcal{L}(s))^n \right).$$

The correspondence between the closed geodesic (or, equivalently, conjugacy classes of hyperbolic elements) and the periodic orbits of  $T$  is particularly simple for Schottky groups and we recall it in the form given in [26] (where it is given in a more complicated setting of co-compact groups):

Closed geodesics on  $\Gamma \backslash \mathbb{H}$ ,  $\gamma$  of length  $l(\gamma)$ , and word length  $|\gamma|$  are in one to one correspondence with periodic orbits  $\{x, Tx, \dots, T^{n-1}x\}$  such that  $\|T^n(x)\| = \exp \ell(\gamma)$ , and  $n = |\gamma|$ . For prime closed geodesics we have the same correspondence with primitive periodic orbits of  $T$ .

It is not needed for us to recall the precise definition of the *word length*. Roughly speaking it is the number of generators of  $\Gamma$  needed to write down  $\gamma$ .

To evaluate  $\text{tr} \mathcal{L}(s)^m$  we write

$$\text{tr} \mathcal{L}(s)^m = \sum_{(i_1, \dots, i_m), i_1 = i_m} \text{tr} (\mathcal{L}_{i_1 i_2} \circ \dots \circ \mathcal{L}_{i_{m-1} i_m}),$$

where in the notation of (3.12) we have

$$\begin{aligned} \mathcal{L}_{i_1 i_2} \circ \dots \circ \mathcal{L}_{i_{m-1} i_m} u(z) &= \|D(f_{i_1 i_2} \circ \dots \circ f_{i_{m-1} i_m})(z)\|^s u(f_{i_1 i_2} \circ \dots \circ f_{i_{m-1} i_m}(z)), \\ f_{i_1 i_2} \circ \dots \circ f_{i_{m-1} i_m} : D_{i_1} &\longrightarrow D_{i_1}. \end{aligned}$$

The trace of this operator is non-zero only if  $f_{i_1 i_2} \circ \dots \circ f_{i_{m-1} i_m}$  has a fixed point in  $D_{i_1}$ . Since this transformation corresponds to an element of  $\Gamma$  that fixed point is unique. Let us call this element  $\gamma^{-1}$ . Since it corresponds to a given periodic point,  $x$ , of  $T^n$ , (corresponding to a fixed point of  $f_{i_1 i_2} \circ \dots \circ f_{i_{m-1} i_m}$ ),  $\gamma$  is determined uniquely by  $x$  and  $n$ :

$$\gamma = \gamma(x, n), \quad T^n x = x.$$

By conjugation and a choice of coordinates  $z$  it can be put into the form

$$\gamma(z) = e^{\ell(\gamma)} (e^{i\theta_1(\gamma)} z_1, \dots, e^{i\theta_n(\gamma)} z_n),$$

and the trace can be evaluated on the Hilbert space  $H^2(B_{\mathbb{C}^n}(0, 1))$ . Using the basis (3.13) we can write the kernel of  $\mathcal{L}_{i_1 i_2} \circ \dots \circ \mathcal{L}_{i_{m-1} i_m}$  as

$$\begin{aligned} \mathcal{L}_{i_1 i_2} \circ \dots \circ \mathcal{L}_{i_{m-1} i_m}(z, w) &= |\gamma'(0)|^{-s} \sum_{\alpha \in \mathbb{N}_0^n} c_\alpha (\gamma^{-1}(z))^\alpha \bar{w}^\alpha \\ &= \sum_{\alpha \in \mathbb{N}_0^n} c_\alpha e^{-(s+|\alpha|)\ell(\gamma) - i\langle \theta(\gamma), \alpha \rangle} z^\alpha \bar{w}^\alpha. \end{aligned}$$

The evaluation of the trace is now clear.

Returning to (3.14), we obtain for  $\operatorname{Re} s$  sufficiently large (using  $\{\gamma\}$ 's to denote the conjugacy classes of primitive elements of  $\Gamma$ ),

$$\begin{aligned} \det(I - z\mathcal{L}(s)) &= \exp \left( - \sum_{n=1}^{\infty} \frac{z^n}{n} \sum_{T^n x=x} \sum_{\alpha \in \mathbb{N}_0^n} e^{-(s+|\alpha|)\ell(\gamma(x,n)) - i\langle \theta(\gamma(x,n)), \alpha \rangle} \right) \\ &= \exp \left( - \sum_{n=1}^{\infty} \sum_{\substack{\{\gamma\} \\ |\gamma|=n}} \sum_{k=1}^{\infty} \frac{z^{nk}}{k} \sum_{\alpha \in \mathbb{N}_0^n} e^{-k((s+|\alpha|)\ell(\gamma) - i\langle \theta(\gamma), \alpha \rangle)} \right) \\ &= \prod_{\{\gamma\}} \prod_{\alpha \in \mathbb{N}_0^n} \left( 1 - z^{|\gamma|} e^{-i\langle \theta(\gamma), \alpha \rangle} e^{-(s+|\alpha|)\ell(\gamma)} \right) \end{aligned}$$

which in view of (3.1) proves the proposition once we put  $z = 1$ .  $\square$

**Remark.** The proof above is inspired by the work on the distribution of resonances in Euclidean scattering - see [34, Proposition 2]. The Fredholm determinant method and the use of Weyl inequalities in the study of resonances were introduced by Melrose [19] and developed further by many authors - see [31], [36], and references given there. That was done at about the same time as David Fried (across the Charles River from Melrose) was applying the Grothendieck-Fredholm theory to multidimensional zeta-functions [4]. In both situation the enemy is the exponential growth for complex energies  $s$ , which is eliminated thanks to analyticity properties of the kernel of the operator.

Finally, we remark that in view of the lower bounds on the number of zeros of  $Z_\gamma$  obtained in [7] in dimension two, and in [21] in general, we see from Proposition 3.4 that the upper bound (3.5) is optimal for any  $\gamma$ .

#### 4. QUASI-SELF-SIMILARITY OF $\Lambda(\Gamma)$ AND MARKOV PARTITIONS FOR $\Gamma \backslash \mathbb{H}^{n+1}$

In this section we will review the results on convex co-compact Schottky groups coming essentially from [32] and [18]. The geometric point of view presented here was explained to the authors by Curt McMullen<sup>3</sup>.

We start with a more general definition of convex co-compact subgroups of  $\operatorname{Isom}(\mathbb{H}^{n+1})$ . A discrete subgroup is called *convex co-compact* if

$$(4.1) \quad \Gamma \backslash C(\Lambda(\Gamma)) \text{ is compact, } C(\Lambda(\Gamma)) \stackrel{\text{def}}{=} \text{convex hull}(\Lambda(\Gamma))$$

Here, the convex hull is meant in the sense of the hyperbolic metric on  $\mathbb{H}^{n+1}$ :  $\Lambda(\Gamma) \subset \partial \mathbb{H}^{n+1}$ , and  $\gamma$  acts on it in the usual way. In particular this implies that  $\Gamma \backslash C(\Lambda(\Gamma))$  has a compact fundamental domain in  $\mathbb{H}^{n+1}$ .

The first result gives a quasi-self-similarity for arbitrary convex co-compact groups:

**Proposition 4.1.** *Suppose that  $\Gamma \subset \operatorname{Isom}(\mathbb{H}^{n+1})$  is convex co-compact in the sense of (4.1). Then there exist  $c > 0$  and  $r_0 > 0$  such that for any  $x_0 \in \Lambda(\Gamma)$  and  $r < r_0$  there exists a map  $g : B_{\mathbb{S}^n}(x_0, r) \rightarrow \mathbb{S}^n$  with the properties*

$$(4.2) \quad \begin{aligned} &g(\Lambda(\Gamma) \cap B_{\mathbb{S}^n}(x_0, r)) \subset \Lambda(\Gamma) \\ &cr^{-1}d_{\mathbb{S}^n}(x, y) \leq d_{\mathbb{S}^n}(g(x), g(y)) \leq c^{-1}r^{-1}d_{\mathbb{S}^n}(x, y), \quad x, y \in B_{\mathbb{S}^n}(x_0, r). \end{aligned}$$

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<sup>3</sup>who also generously provided us with Figures 4,5, and previously with the essential part of Figure 3.

*Proof.* We proceed following the argument in [32, Sect.3]. Let us fix  $z_0 \in C(\Lambda(\Gamma))$ . If  $L$  is the geodesic ray through  $z_0$  and  $x_0$ , then

$$\exists C > 0 \forall z \in L \exists \gamma \in \Gamma \quad d(\gamma^{-1}z_0, z) < C.$$

This follows from the compactness of  $\Gamma \backslash C(\Lambda(\Gamma))$ : for any point on the ray,  $z$ , there exists an element of the orbit of  $z_0$  within a finite distance from  $z$ . We can now choose  $z = z(r)$  on the ray  $L$  so that  $d(z, z_0) = \log(1/r)$ , and then  $\gamma$  such that  $d(\gamma^{-1}z_0, z) = \log(1/r) + \mathcal{O}(1)$ .

If  $x_\gamma$  is the end point of the geodesic ray through  $z_0$  and  $\gamma^{-1}z_0$ , then for a fixed  $C_1$ , the ball  $B_{\mathbb{S}^n}(x_\gamma, C_1 r)$  covers  $B_{\mathbb{S}^n}(x_0, r)$ . The action of  $\gamma$  on  $B_{\mathbb{S}^n}(x_\gamma, C_1 r)$  satisfies (4.2):  $\Lambda(\Gamma)$  is  $\Gamma$ -invariant, and the other property follows by putting  $\gamma$  into the normal form (2.1). Since  $z_0$  was fixed and we have no dependence on  $\gamma$ , the proof is completed.  $\square$

Strictly speaking we will not use the quasi-self-similarity explicitly. It is present implicitly in the proof of Proposition 4.2 below where again the compactness of  $\Gamma \backslash C\Lambda(\Gamma)$  is crucial. We remark that for  $n = 1$ , that is for surfaces, Proposition 4.1 shows that, in the upper half plane model chosen so that  $\Lambda \in \mathbb{R}$ ,

$$\Lambda(\Gamma) + [-h, h] = \bigcup_{j=1}^{N(h)} [a_j(h), b_j(h)], \quad b_j(h) < a_{j+1}(h), \quad |b_j(h) - a_j(h)| \leq Kh,$$

with  $K$  independent of  $h$ . That gives a simple proof of our main result – see Section 5.

For the general we recall the definition of a Markov partition in a simple form [18] applicable here. Thus let  $\mu_\Lambda$  be the  $\delta$ -Hausdorff measure restricted to  $\Lambda(\Gamma)$ . Let  $T$  and  $\mathcal{D}$  be defined as in (3.2).

A *Markov partition* for the Schottky group  $\Gamma$  is given by a collection,  $\mathcal{P}$ , of subsets of  $\mathcal{D} \subset \mathbb{S}^n$ , satisfying

$$\begin{aligned} \forall P, Q \in \mathcal{P} \quad \mu_\Lambda(T(P) \cap Q) \neq 0 &\implies T(P) \supset Q \\ T \text{ is a homeomorphism on a neighbourhood of } P \cap (T|_P)^{-1}(Q) &\text{ if } \mu_\Lambda(T(P) \cap Q) \neq 0 \\ \forall P \in \mathcal{P} \quad \mu_\Lambda(P) &> 0 \\ \forall P, Q \in \mathcal{P}, P \neq Q \quad \mu_\Lambda(P \cap Q) &= 0 \\ \mu_\Lambda(T(P)) &= \sum_{\substack{Q \in \mathcal{P} \\ \mu_\Lambda(T(P) \cap Q) \neq 0}} \mu_\Lambda(Q). \end{aligned} \tag{4.3}$$

A refinement of  $\mathcal{P}$  is a new Markov partition given by

$$\mathcal{R}(\mathcal{P}) \stackrel{\text{def}}{=} \{(T|_P)^{-1}(Q) : P, Q \in \mathcal{P}, \mu_\Lambda(T(P) \cap Q) \neq 0\},$$

see Fig.5. The refinements can of course be iterated giving  $\mathcal{R}^m(\mathcal{P})$ .

The Markov partition is said to be expanding if there exists a metric  $\|\cdot\|$  on  $\mathcal{D}$  and a real  $\theta > 1$  such that  $\|DT(v)\xi\| \geq \theta\|\xi\|, v \in V, \xi \in T_v V$ . An example of a Markov partition for the Schottky group map is given by

$$\mathcal{P}_0 = \{\mathcal{D}_i : 1 \leq i \leq 2\ell\}.$$

If  $N$  is big enough its refinement,  $\mathcal{D}^N = \mathcal{R}^N(\mathcal{P})$ , considered in Section 4, is expanding.

The result we need to estimate the zeta function is given in the following

**Proposition 4.2.** *Let  $\Gamma$  be a convex co-compact Schottky group as defined in Section 2. Then there exist positive constants  $h_0$ ,  $C_0$ , and  $C_1$ , such that for  $0 < h < h_0$  there exists a Markov*

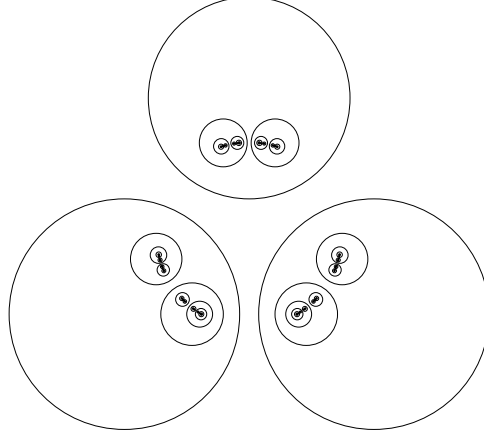


FIGURE 5. A Markov partition and its refinements for the reflection group  $\Gamma_\theta$  of Fig. 1 (considered here as a Kleinian conformal group on  $\partial\mathbb{H}^3 \simeq \mathbb{R}^2 \cup \infty$ ).

partition,  $\mathcal{P}(h)$ , for  $\Gamma$  with the following properties:

$$(4.5) \quad \begin{aligned} & \forall P \in \mathcal{P}(h) \exists x_P \in \mathbb{S}^n, \quad P \subset B_{\mathbb{S}^n}(x_P, C_0 h) \\ & \forall P, Q \in \mathcal{P}(h) \quad \Lambda(\Gamma) \cap T(P) \cap Q \neq \emptyset \implies d_{\mathbb{S}^n}((T|_P)^{-1}(Q), \partial P) > h/C_1. \end{aligned}$$

In fact, we can make each element of  $\mathcal{P}(h)$  a union of a fixed number of balls of radii comparable to  $h$ .

*Proof.* Rather than use the successive refinements of the Markov partition  $\mathcal{D}^N$  (which would not work) we will apply the following geometric observation: the projection of any geodesic in  $C(\Lambda(\Gamma))$  to  $\Gamma \backslash \mathbb{H}^{n+1}$  intersects the projections of  $C(\partial\mathcal{D}_l)$  at uniformly bounded intervals, say by  $d_0 > 0$ . This follows from the bound on the diameter of the compact fundamental domain of  $\Gamma \backslash C(\Lambda(\Gamma))$ .

For  $e^{-m} < h < e^{-m+1}$  we choose  $\mathcal{P}(h)$  in three steps. Let  $z_0 \in \mathbb{H}^{n+1}$  be a fixed point, and  $\mathcal{P}_0$  the partition  $\mathcal{D}^N$ . We then put

$$\begin{aligned} \mathcal{P}_k(h) &= \{P \in \mathcal{R}^k(\mathcal{P}_0) : m < d_{\mathbb{H}^{n+1}}(C(P), z_0) \leq m + d_0\} \\ \tilde{\mathcal{P}}(h) &\stackrel{\text{def}}{=} \bigcup_{k \geq 1} \{P \in \mathcal{P}_k(h) : P \not\subset Q \in \mathcal{P}_{k-1}(h)\}. \end{aligned}$$

The choice of  $d_0$  shows that the elements of  $\tilde{\mathcal{P}}(h)$  cover  $\Lambda(\Gamma)$ . From  $d(C(P), z_0) = m + \mathcal{O}(1)$ , we conclude that  $P \subset B_{\mathbb{S}^n}(x_P, C_0 h)$ ,  $C_0$  uniformly bounded,  $h \sim e^{-m}$ .

To obtain the strong form of the Markov partition property given in the second part of (4.5) we modify  $\tilde{\mathcal{P}}(h)$  as follows:

$$(4.6) \quad \hat{\mathcal{P}}(h) = \{\{x \in \mathbb{S}^n : d(x, \Lambda(\Gamma) \cap P) < h/C_1\} : P \in \tilde{\mathcal{P}}(h)\}.$$

We still have the property that

$$\forall P, Q \in \hat{\mathcal{P}}(h) \quad P \neq Q \implies P \cap Q \cap \Lambda(\Gamma) = \emptyset.$$

If  $\widehat{\mathcal{P}}(h) \ni Q \subset D_j$ , where  $D_j$ 's are as (3.12) then  $f_{ij}$  is a contraction of  $Q$  into a  $\widehat{\mathcal{P}}(h) \ni Q' \subset D_i$ . In fact,  $f_{ij}$  is a contraction of  $D_j$  into  $D_i$  and it preserves  $\Lambda(\Gamma)$ . That shows the second part of (4.5).

Depending on the contraction constants of  $f_{ij}$ 's (a finite set) we can replace elements of  $\widehat{\mathcal{P}}(h)$  with unions of a fixed number of balls (the number depending only on the constants  $C_0$  and  $C_1$  and the dimension) so that all the properties still hold.  $\square$

## 5. ESTIMATES IN TERMS OF THE DIMENSION OF $\Lambda(\Gamma)$ .

In Proposition 3.4 we used the Markov partition (4.4). It is clear from the proof that a Markov partition for which (4.5) holds (for some fixed  $h$ ) can be used instead as long as the contraction property is valid in some complex neighbourhood of the components of the partition. For the proof of Theorem stated in Section 1 we will modify  $D_j$ 's in the definition of  $\mathcal{L}(s)$  in a way dependent on the size of  $s$ : we will use Proposition (4.2) with  $h = 1/|s|$ . The self-similarity structure of  $\Lambda(\Gamma)$  will show that we can choose  $D_j = D_j(h)$  to be a union of  $\mathcal{O}(h^{-\delta})$  disjoint balls of radii  $\sim h$ . A modification of the argument used in the proof of Proposition 3.2 will then give (1.2).

We start with the following lemma which is a more precise version of the argument already used in the proof of Proposition 3.2:

**Lemma 5.1.** *Suppose that  $\Omega_j \subset \mathbb{C}^n$ ,  $j = 1, 2$ , are open sets, and  $\Omega_1 = \bigcup_{k=1}^K B_{\mathbb{C}^n}(z_k, r_k)$ . Let  $g$  be a holomorphic mapping defined on a neighbourhood,  $\widetilde{\Omega}_1$  of  $\Omega_1$  with values in  $\Omega_2$ , satisfying*

$$d_{\mathbb{C}^n}(g(\Omega_1), \partial\Omega_2) > 1/C_0 > 0.$$

If

$$A : H^2(\Omega_2) \longrightarrow H^2(\Omega_1), \quad Au(z) \stackrel{\text{def}}{=} u(g(z)),$$

then for some  $C_1$  depending only on  $r_k$ 's,  $K$ ,  $d_{\mathbb{C}^n}(\Omega_1, \partial\Omega_2)$ , and  $\min_{\widetilde{\Omega}_1} \|Dg\|_{\mathbb{C}^n \rightarrow \mathbb{C}^n}$ , we have

$$\mu_\ell(A) \leq C_1 e^{-\ell^{1/n}/C_1},$$

where  $\mu_\ell(A)$ 's are the characteristic values of  $A$ .

*Proof.* We define a new Hilbert space

$$\mathcal{H} \stackrel{\text{def}}{=} \bigoplus_{k=1}^K H^2(B_k), \quad B_k = B_{\mathbb{C}^n}(z_k, r_k),$$

and a natural operator

$$J : H^2(\Omega_1) \longrightarrow \mathcal{H}, \quad (Ju)_k = u|_{B_k}.$$

We easily check that  $J^*J : H^2(\Omega_1) \rightarrow H^2(\Omega_1)$  is invertible with constants depending only on  $K$ . Hence

$$\mu_\ell(A) = \mu_\ell((J^*J)^{-1}J^*JA) \leq \|(J^*J)^{-1}\| \|J^*\| \mu_\ell(JA).$$

We then notice that

$$\mu_{k\ell}(JA) \leq k \max_{1 \leq k \leq K} \mu_\ell(A_k),$$

where

$$A_k : H^2(\Omega_2) \longrightarrow H^2(B_k), \quad A_k u(z) = u(g_k(z)), \quad g_k = g|_{B_k}.$$



To estimate the characteristic values of  $A_k$  we observe that we can extend  $g_k$  to a larger ball,  $\tilde{B}_k$  (contained in  $\tilde{\Omega}_1$ ) and such that the image of its closure still lies in  $\Omega_2$  (since we know that  $\min_{\tilde{\Omega}_1} \|Dg\|_{\mathbb{C}^n \rightarrow \mathbb{C}^n}$  is strictly less than 1). That gives us the operators  $R_k : H^2(\tilde{B}_k) \rightarrow H^2(B_k)$ ,  $R_k u = u|_{B_k}$ , and  $\tilde{A}_k$  defined as  $A_k$  but with  $B_k$  replaced by  $\tilde{B}_k$ . We now have  $A_k = R_k \tilde{A}_k$  and consequently,

$$\mu_\ell(A_k) \leq \|\tilde{A}_k\| \mu_\ell(R_k).$$

Lemma 3.3 gives  $\mu_\ell(R_k) \leq C_2 \exp(-\ell^{1/n}/C_2)$  completing the proof.  $\square$

*Proof of Theorem.* As outlined in the beginning of the section we put  $h = 1/|s|$ , where  $|s|$  is large but  $|\operatorname{Re} s|$  is uniformly bounded. In Proposition 4.2 each element of the Markov partition is given as union of (a fixed number of) balls. We can complexify this set by taking a corresponding union of balls in  $\mathbb{C}^n$  and all the properties hold for the analytic continuations of  $f_{ij}$ 's defined in (3.12).

The now classical results of Patterson and Sullivan [32] on the dimension of the limit set show that the total number of the balls is  $\mathcal{O}(h^{-\delta})$ : what we are using here is the fact that the Hausdorff measure of  $\Lambda(\Gamma)$  is finite.

We can now apply the same procedure as in the proof of Proposition 3.2 using Lemma 5.1. What we have gained is a bound on the weight: since  $|\operatorname{Re} s| \leq C$  and  $f'_{ij}$  is real on the real  $\mathbb{S}^n$

$$|[f'_{ij}(z)]^s| \leq C \exp(|s| |\arg f'_{ij}(z)|) \leq C \exp(C_1 |s| |\operatorname{Im} z|) \leq C_2, \quad z \in D_j(h).$$

We write  $\mathcal{L}(s)$  as a sum of fixed number of operators  $\mathcal{L}_{ij}(s)$  each of which is a direct sum of  $\mathcal{O}(h^{-\delta})$  operators. The balls and contractions are uniform after rescaling by  $h$  and hence the characteristic values of each of these operators satisfy the bound  $\mu_\ell \leq C\gamma^\ell$ ,  $0 < \gamma < 1$ . Using (3.9) and (3.11) we obtain the bound

$$\log |\det(I - \mathcal{L}(s))| \leq CP(h) = \mathcal{O}(h^{-\delta}),$$

and this is (1.2).  $\square$

*Proof of Corollary 1.* The definition of  $Z_\Gamma(s)$  (3.1) shows that for  $\operatorname{Re} s > C_1$  we have  $|Z_\Gamma(s)| > 1/2$ . The Jensen formula then shows that the left hand side of (1.3) is bounded by

$$\sum \{m_\Gamma(s) : |s - ir - C_1| \leq C_2\} \leq 2 \max_{\substack{|s| \leq r + C_3 \\ |\operatorname{Re} s| \leq C_0}} \log |Z_\Gamma(s)| + C_4,$$

and (1.3) follows from (1.2).  $\square$

## 6. SCHOTTKY MANIFOLDS AND RESONANCES

We recall that a complete Riemannian manifold of constant curvature  $-1$  is said to be Schottky if its fundamental group is Schottky. In low dimensions Schottky manifolds can be described geometrically.

**Proposition 6.1.** *Any convex co-compact hyperbolic surface is Schottky.*

This result is proved by Button [2] and for the reader's convenience we sketch the proof.

*Proof.* Any convex co-compact surface is topologically described by two integers  $(g, f)$ : its numbers  $g$  of holes and  $f$  of funnels, with the conditions  $g \geq 0$ ,  $f \geq 1$  and  $f \geq 3$  if  $g = 0$ . For any such pair  $(g, f)$ , there does exist a Schottky surface of this type and we choose for each type  $(g, f)$  such a surface  $M_{g,f}$ . The projection onto  $M_{g,f}$  of the boundary of the Schottky domain is a collection  $\mathcal{L}_1, \dots, \mathcal{L}_\ell$ , of mutually disjoint geodesic lines.

Let  $M$  be any hyperbolic convex co-compact surface. The surface  $M$  is homeomorphic to some  $M_{g,f}$ . Pushing back on  $M$  the geodesic lines  $\mathcal{L}_i, i = 1, \dots, \ell$  of  $M_{g,f}$  and cutting  $M$  along these curves, we obtain in the hyperbolic plane a domain whose boundary is the union of paired mutually disjoint curves  $\mathcal{C}_i, \mathcal{C}_{\ell+i}, i = 1, \dots, \ell$ , each one with a pair of points at infinity. These points pair determine intervals, which are mutually disjoint (the curves  $\mathcal{C}_j, j = 1, \dots, 2\ell$  don't intersect). The intervals are paired with an hyperbolic transformation, so give a Schottky group, which coincide with the fundamental group of the surface  $M$ .  $\square$

*Proof of Corollary 2.* For a Schottky manifold, the fundamental groups is Schottky, and hence,  $M = \Gamma \backslash \mathbb{H}^{n+1}, \Gamma \subset \text{Isom}^+(\mathbb{H}^{n+1})$ . We then introduce its zeta function  $Z_M$  as the zeta function  $Z_\Gamma$  of the group  $\Gamma$ . Following Patterson and Perry [20] we introduce the spectral sets  $\mathcal{P}_M$  and  $\mathcal{S}_M$  defined by the Laplace-Beltrami operator  $\Delta_M$  on  $M$ :

$$\begin{aligned} \mathcal{P}_M &= \{s : \text{Re } s > n/2, s(n-s) \text{ is a } L^2 \text{ eigenvalue of } \Delta_M\} \\ \mathcal{S}_M &= \{s : \text{Re } s < n/2, s \text{ is a singularity of the scattering matrix } S_M\} \end{aligned}$$

Moreover, each complex  $s$  in  $\mathcal{P}_M$  has a multiplicity denoted by  $m_M(s)$ , each  $s$  in  $\mathcal{S}_M$  a pole multiplicity denoted by  $m_M^-(s)$ . In the case of surfaces, the divisor of the Selberg zeta function  $Z_M$  is given by the following formula:

$$-\chi_M \sum_{k=0}^{\infty} (2k+1)[-k] + m_M\left(\frac{n}{2}\right) \left[\frac{n}{2}\right] + \sum_{s \in \mathcal{P}_M} m_M(s)[s] + \sum_{s \in \mathcal{S}_M} m_M^-(s)[s],$$

where  $\chi_M$  is the Euler-characteristic of  $M$ , see [20, Theorem 1.2] The zeta function,  $Z_M$ , is entire and in any half-plane  $\{\text{Re } s > -C_0\}$ , and the formula above shows that the bounds on the number of its zeros provide bounds on the number of resonances. The dimension of the limit set,  $\delta$  depends only on  $\Gamma$  and, as shown in [32], it gives the Hausdorff dimension of the recurrent set for the geodesic flow on  $T^*M$  by the formula  $2(1 + \delta)$ .  $\square$

For a convex co-compact hyperbolic manifold  $M$ , Patterson and Perry give a formula for the divisor of the zeta function  $Z_M$  in any (even) dimension, but it does not imply (in the non-Schottky case) that the zeta function is entire. In the case of Schottky groups, the zeta function  $Z_M$  is entire, as it was shown in the Proposition 3.4. Hence we concluded that Corollary 2 holds also for Schottky manifold.

We conclude with some remarks about Kleinian groups in dimension  $n + 1 = 3$ . Schottky 3-manifold are geometrically described by Maskit [14]:

**Proposition 6.2.** *A hyperbolic convex co-compact, non compact 3-manifold is Schottky if and only if its fundamental group is a free group of finite type.*

While non compact surfaces of finite geometric type have always a free fundamental group, that is not true for 3-manifold. For instance, if  $\Gamma$  is a co-compact surface group, the 3-manifold  $\mathbb{H}^3/\Gamma$  is convex co-compact with a non-free fundamental group. Quasi-fuchsian groups (that is, deformation of such a  $\Gamma$  in  $\text{Isom}(\mathbb{H}^3)$ ) give similar examples.

Finally, we note that the bound on the number of zeros of  $Z_\Gamma$  established here for Schottky groups is valid for any group  $\Gamma$ , for which a expanding Markov partition can be built. Anderson and Rocha [1] construct such Markov partition for any function group. This class of groups does not exhaust all convex co-compact groups (the complement in the 3-sphere of a regular neighbourhood of a graph is not in this class) and it is not known if all convex co-compact Kleinian groups admit an expanding Markov partition.

$\theta$	dim
$10^\circ$	0.11600945
$20^\circ$	0.15118368
$30^\circ$	0.18398306
$40^\circ$	0.21776581

TABLE 1. Dimensions of the limit set for relevant values of  $\theta$ .

## 7. NUMERICAL RESULTS.

**7.1. Discussion.** In this section, we present numerical results on the distribution of zeros of a closely related dynamical zeta function  $Z(s)$  for a conformal dynamical system. We consider the simple case of groups  $\Gamma_\theta$  generated by reflections in three symmetrically placed circles perpendicular to the unit circle (considered as the boundary of the Poincaré disc), and intersecting it at angles  $\theta$  – see Fig.3 where  $\theta = 110^\circ$ .

Numerical computations of the zeta function in that case have been already performed by McMullen [18] and Jenkinson-Pollicott [8]. Their goal was to find an efficient way of computing the dimension of limit sets. Table 1 gives the (approximate) dimensions of the limit sets for the relevant angles, calculated as the largest real zero of  $Z(s)$  [8] using Newton’s method.

Figures 7-9 show

$$(7.1) \quad \frac{\log(|\{s \in [x_0, x_1] \times [y_0, y] : Z(s) = 0\}|)}{\log(y)} - 1$$

as a function of  $y$ . In each plot, the value of  $x_0$  is varied to test the dependence of the distribution on the region in which we count: The blue line corresponds to  $x_0 = -0.2$ , the red line  $x_0 = -0.1$ , and the black line  $x_0 = +0.1$ . The data show that most of the zeros lie in the left half plane. Based on the theorems proved in earlier sections, we expect the curves to be bounded above by the dimension (the thick blue line) asymptotically. This is not the case, except for the black line, which represents zeros with  $\operatorname{Re}(s) > x_0 = +0.1$ .

Similarly, Figures 10-12 show  $\frac{\log(\log(|Z(s)|))}{\log(|s|)}$  as a function of  $|s|$ , for a large number of points in the rectangle  $[-0.2, 1.0] \times [0, 10^3]$ . In this case, we also expect the curves to be asymptotically bounded by the dimension. This is also not the case. The only reasonable explanation, barring errors in the numerical calculations, is that the asymptotic upper bound is accurate only for very large values of  $\operatorname{Im}(s)$ , and we were not able to calculate  $Z(s)$  reliably for such values. These results also show that  $Z(s)$  has plenty of zeros in regions of interest.

**7.2. Implementation notes.** To count the number of zeros of  $Z(s)$  in a given region  $\Omega$  in the complex plane, we rely on the Argument Principle:

$$(7.2) \quad |\{s \in \Omega : Z(s) = 0\}| = \frac{1}{2\pi i} \int_{\partial\Omega} \frac{Z'(s)}{Z(s)} ds.$$

To evaluate  $Z(s)$ , our main technical tool comes from Jenkinson and Pollicott [8], though we note that the essential ideas were used in Eckhardt, et. al. [3] and date back to Ruelle [29].

First, some notation: Let us denote symbolic sequences of length  $|\gamma| = n$  by  $\gamma$ . That is,  $\gamma = (\gamma(0), \gamma(1), \dots, \gamma(n))$ ,  $\gamma(k) \in \{0, 1, 2\}$ , and  $\gamma(0) = \gamma(n)$ . Such sequences represent periodic orbits of length  $n$ . To each such sequence  $\gamma$  we associate a composition of reflections  $\phi_\gamma = \phi_{\gamma(n)} \circ \dots \circ \phi_{\gamma(1)} : D_{\gamma(0)} \rightarrow D_{\gamma(0)}$ . As  $\phi_\gamma$  is a contraction of  $D_{\gamma(0)}$  into itself, it has a unique fixed point  $z_\gamma$ .

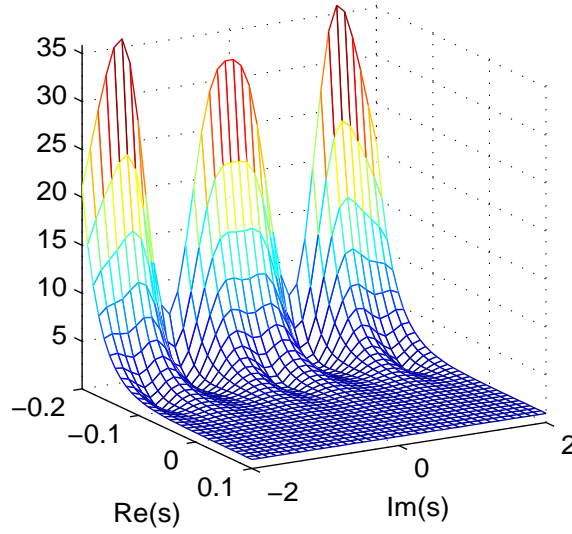


FIGURE 6. This plot shows  $|Z(s)|$  in the square  $[-0.2, 0.3] \times [-2, 2]$ , for  $\theta = 30^\circ$ .

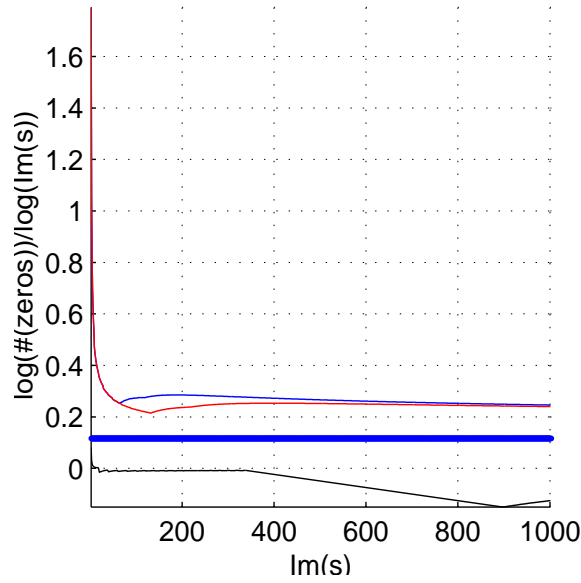


FIGURE 7. This plot shows  $\frac{\log(|\{s \in [x_0, x_1] \times [y_0, y] : Z(s) = 0\}|)}{\log(y)} - 1$  as a function of  $y$ , for different values of  $x_0$ : The thin blue line is for  $x_0 = -0.2$ , the red line for  $x_0 = -0.1$ , and the black line for  $x_0 = +0.1$ . Note that the value of  $x_1$  is not very important because  $Z(s)$  decays very rapidly for large  $\text{Re}(s)$ . Thus, we set  $x_1 = 10$  throughout. The value of  $y_0$  is fixed at  $-0.1$ , to avoid integrating over any zeros. The thick horizontal line indicates the dimension of the corresponding limit set. In this plot,  $\theta = 10^\circ$ .

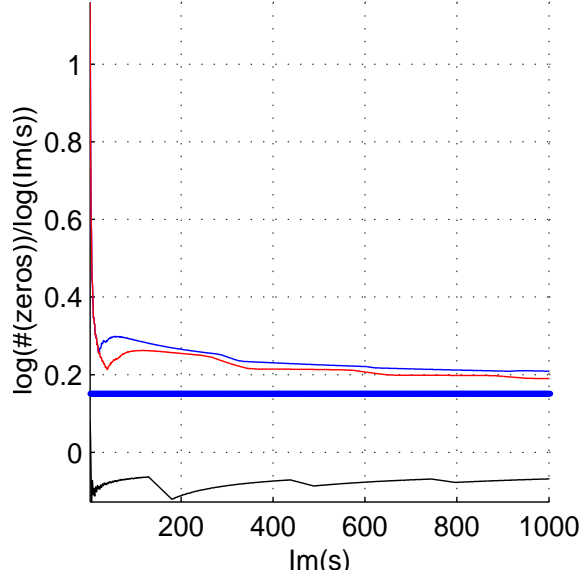


FIGURE 8. This plot shows  $\frac{\log(|\{s \in [x_0, x_1] \times [y_0, y] : Z(s) = 0\}|)}{\log(y)} - 1$  as a function of  $y$ , for different values of  $x_0$ : The thin blue line is for  $x_0 = -0.2$ , the red line for  $x_0 = -0.1$ , and the black line for  $x_0 = +0.1$ . Note that the value of  $x_1$  is not very important because  $Z(s)$  decays very rapidly for large  $\text{Re}(s)$ . Thus, we set  $x_1 = 10$  throughout. The value of  $y_0$  is fixed at  $-0.1$ , to avoid integrating over any zeros. The thick horizontal line indicates the dimension of the corresponding limit set. In this plot,  $\theta = 20^\circ$ .

It is shown in Jenkinson and Pollicott that  $Z(s) = \lim_{M \rightarrow \infty} Z_M(s)$ , where

$$(7.3) \quad Z_M(s) = 1 + \sum_{N=1}^M \sum_{r=1}^N \frac{(-1)^r}{r!} \sum_{n \in P(N,r)} \prod_{k=1}^r \frac{1}{n_k} \sum_{|\gamma|=n_k} \frac{|\phi'_\gamma(z_\gamma)|^s}{|1 - \phi'_\gamma(z_\gamma)|^2},$$

where  $P(N, r)$  is the set of all  $r$ -tuples of positive integers  $(n_1, \dots, n_r)$  such that  $n_1 + \dots + n_r = N$ . The series (in  $N$ ) converges absolutely in  $\{s : \text{Re}(s) > -a\}$  for some positive  $a$ .

Equation (7.3) lets us evaluate  $Z(s)$  for reasonable values of  $s$  in a straightforward manner. However, we found two simple but useful observations during the course of this calculation:

1. Define

$$(7.4) \quad a_n(s) = \frac{1}{n} \sum_{|\gamma|=n} \frac{|\phi'_\gamma(z_\gamma)|^s}{|1 - \phi'_\gamma(z_\gamma)|^2}$$

and

$$(7.5) \quad B_{N,r}(s) = \frac{(-1)^r}{r!} \sum_{n \in P(N,r)} \prod_{k=1}^r a_{n_k}(s).$$

Then the recursion relation

$$(7.6) \quad B_{N,r}(s) = -\frac{1}{r} \sum_n B_{N-n,r-1}(s) \cdot a_n(s),$$

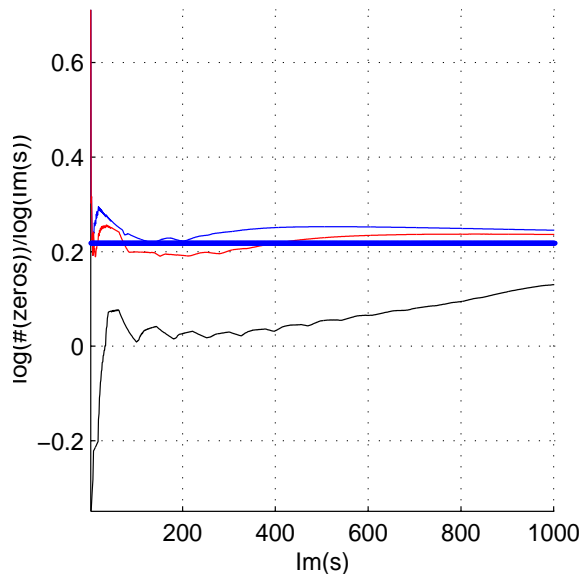


FIGURE 9. This plot shows  $\frac{\log(|\{s \in [x_0, x_1] \times [y_0, y_1] : Z(s) = 0\}|)}{\log(y)} - 1$  as a function of  $y$ , for different values of  $x_0$ : The thin blue line is for  $x_0 = -0.2$ , the red line for  $x_0 = -0.1$ , and the black line for  $x_0 = +0.1$ . Note that the value of  $x_1$  is not very important because  $Z(s)$  decays very rapidly for large  $\text{Re}(s)$ . Thus, we set  $x_1 = 10$  throughout. The value of  $y_0$  is fixed at  $-0.1$ , to avoid integrating over any zeros. The thick horizontal line indicates the dimension of the corresponding limit set. In this plot,  $\theta = 40^\circ$ .

with initial conditions  $B_{N,1}(s) = a_N(s)$ , provides an efficient way to evaluate the sum in (7.3). A similar relation can be derived for  $Z'(s)$  by differentiation.

2. Recall that the maps  $\phi_\gamma$  are compositions of linear fractional transformations. Identifying such maps with matrices in  $GL(2, \mathbb{R})$  in the usual way, we can compute the numbers  $\phi'_\gamma(z_\gamma)$  via matrix multiplications. However, such matrix multiplications can become numerically unstable for larger values of  $|\gamma|$ .

An alternative involves the observation that the matrices  $A_\gamma = A_{\gamma(n)} \cdot \dots \cdot A_{\gamma(1)}$  corresponding to the maps  $\phi_\gamma$  have distinct nonzero real eigenvalues. Let us denote these eigenvalues by  $\lambda_+$  and  $\lambda_-$  so that  $|\lambda_+| > |\lambda_-|$ . Then a simple calculation shows that  $\phi'_\gamma(z_\gamma) = \lambda_-/\lambda_+$ . This becomes simply  $(-1)^{|\gamma|}/\lambda_+^2$  if we normalize the determinants of the generators  $A_0$ ,  $A_1$ , and  $A_2$ . The larger eigenvalue  $\lambda_+$  can be easily computed using a naive power method:

- (a) Choose a random  $v_0$ .
  - (b) For each  $k \geq 0$ , set  $v_{k+1} = A_\gamma v_k / \|A_\gamma v_k\|$  and  $\lambda_+^{(k)} = (A_\gamma v_k, v_k)$ .
  - (c) Iterate until the sequence  $(\lambda_+^{(k)})$  converges, up to some prespecified error tolerance.
- The resulting algorithm is slightly less efficient than direct matrix multiplication, but it is much less susceptible to the effects of round-off error.

Note that it is certainly possible, even desirable, to apply to this problem modern linear algebraic techniques, such as those implemented in ARPACK [10]. But, we found that the power method suffices in these calculations.

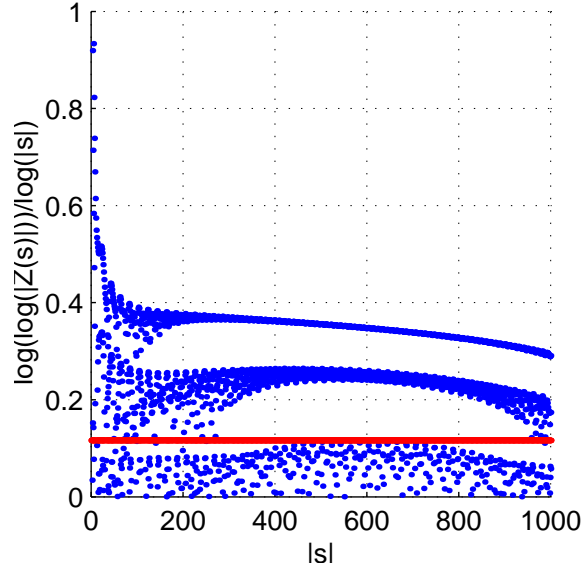


FIGURE 10. This plot shows  $\log(\log(|Z(s)|))/\log(|s|)$  for a large number of points in the rectangle  $[-0.2, 1] \times [0, 10^3]$ . The horizontal line indicates dimension. Here,  $\theta = 10^\circ$ .

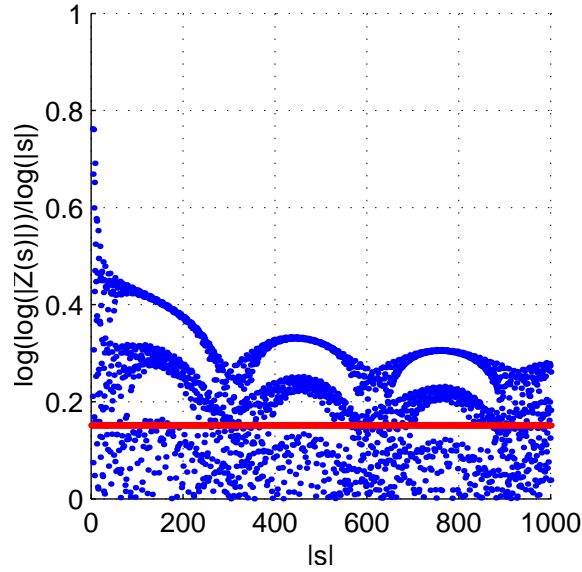


FIGURE 11. This plot shows  $\log(\log(|Z(s)|))/\log(|s|)$  for a large number of points in the rectangle  $[-0.2, 1] \times [0, 10^3]$ . The horizontal line indicates dimension. Here,  $\theta = 20^\circ$ .

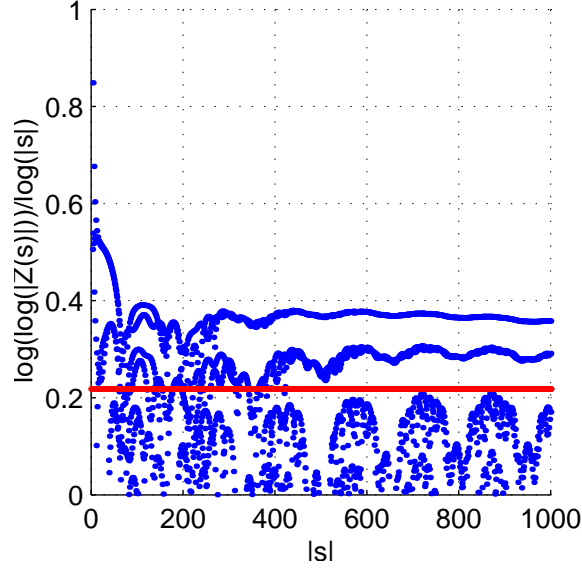


FIGURE 12. This plot shows  $\log(\log(|Z(s)|))/\log(|s|)$  for a large number of points in the rectangle  $[-0.2, 1] \times [0, 10^3]$ . The horizontal line indicates dimension. Here,  $\theta = 40^\circ$ .

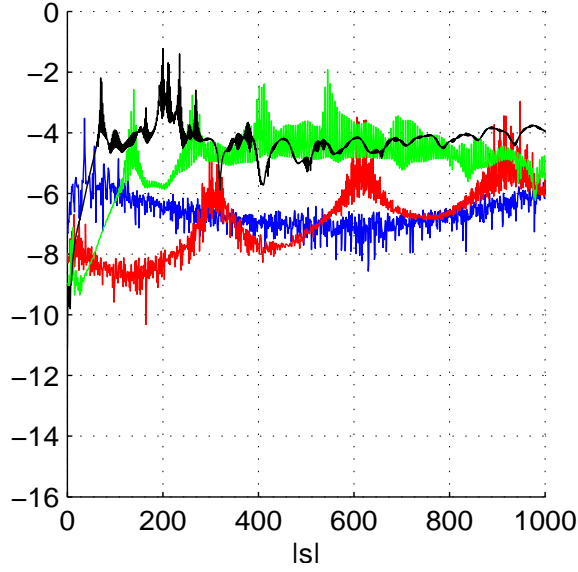


FIGURE 13. Logarithmic plot (base 10) of the modified relative error  $\frac{|R_{12}(s) - R_{13}(s)|}{1 + |R_{12}(s)| + |R_{13}(s)|}$  along the line  $\operatorname{Re}(s) = -0.2$ , where  $R_N(s) = Z'_N(s)/Z_N(s)$ . The blue curve is  $\theta = 10^\circ$ , the red curve  $\theta = 20^\circ$ , the green curve  $\theta = 30^\circ$ , and the black curve  $\theta = 40^\circ$ .



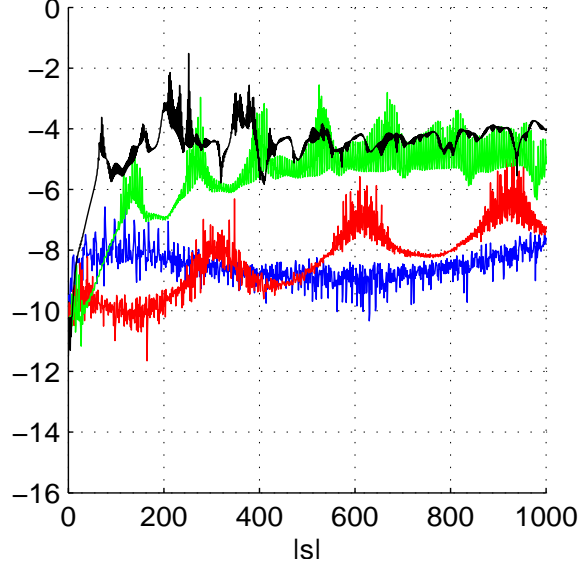


FIGURE 14. Logarithmic plot (base 10) of the modified relative error  $\frac{|R_{12}(s) - R_{13}(s)|}{1 + |R_{12}(s)| + |R_{13}(s)|}$  along the line  $\operatorname{Re}(s) = -0.1$ , where  $R_N(s) = Z'_N(s)/Z_N(s)$ . The blue curve is  $\theta = 10^\circ$ , the red curve  $\theta = 20^\circ$ , the green curve  $\theta = 30^\circ$ , and the black curve  $\theta = 40^\circ$ .

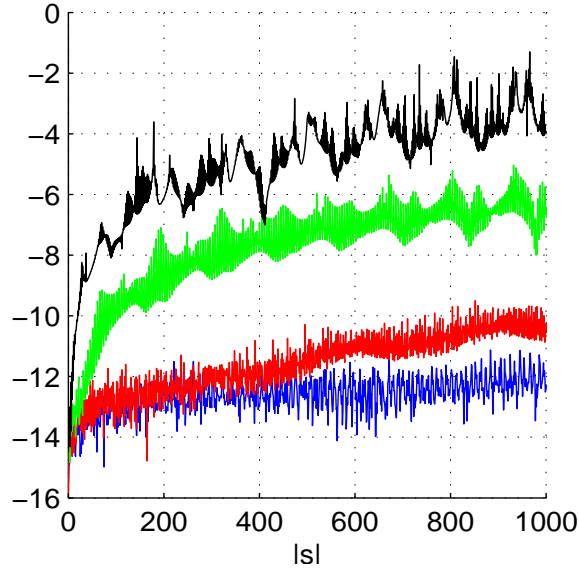


FIGURE 15. Logarithmic plot (base 10) of the modified relative error  $\frac{|R_{12}(s) - R_{13}(s)|}{1 + |R_{12}(s)| + |R_{13}(s)|}$  along the line  $\operatorname{Re}(s) = +0.1$ , where  $R_N(s) = Z'_N(s)/Z_N(s)$ . The blue curve is  $\theta = 10^\circ$ , the red curve  $\theta = 20^\circ$ , the green curve  $\theta = 30^\circ$ , and the black curve  $\theta = 40^\circ$ .

These two simple observations lets us calculate the values of  $Z(s)$  for a wide range of values in an efficient manner. When combined with adaptive gaussian quadrature, Equation (7.3) allows us to evaluate the relevant contour integrals.

Remarks.

1. To calculate the Selberg zeta function  $Z_2(s)$  for closed geodesics on the quotient space  $\Gamma \backslash \mathbb{H}^2$ , we simply sum over periodic orbits of even length, and additionally use  $a_2(n, s) = 2a(n, s)$  instead of  $a(n, s)$  in the recursion relations above. This counts the number of equivalence classes of orbits correctly.
2. The work of Pollicott and Rocha [27] revolves around a closely-related trace formula:

$$(7.7) \quad Z(s) = 1 + \sum_{N=1}^{\infty} \sum_{r=1}^N (-1)^r \sum_{\{[\gamma_1], \dots, [\gamma_r]\} \in P_{\Gamma}(N, r)} \prod_{k=1}^r \frac{|\phi'_{\gamma_k}(z_{\gamma_k})|^s}{|1 - \phi'_{\gamma_k}(z_{\gamma_k})|^2}$$

where  $P_{\Gamma}(N, r) = \{\{[\gamma_1], \dots, [\gamma_r]\} : |\gamma_1| + \dots + |\gamma_r| = N, \gamma_k \text{ primitive}\}$ , and  $[\gamma]$  is the equivalence class of  $\gamma$  under shifts. The primary difference between (7.3) and (7.7) is that the latter sums over sets of equivalence classes of primitive periodic orbits (equivalent up to shifts), whereas the former sums over all periodic orbits. While it is possible to enumerate primitive periodic orbits efficiently,<sup>4</sup> Equation (7.3) still provides a better numerical algorithm, as it is easier to implement and results in faster and more stable code.

**7.3. Error analysis.** Figures 13-15 show the logarithms (base 10) of the modified relative errors

$$(7.8) \quad \frac{|R_{12}(s) - R_{13}(s)|}{1 + |R_{12}(s)| + |R_{13}(s)|},$$

on the lines  $x_0 + i[0, 10^3]$ , for  $x_0 \in \{-0.2, -0.1, 0.1\}$  and where  $R_N(s) = Z'_N(s)/Z_N(s)$ . This formula interpolates between the absolute and the relative errors, and measures the convergence of the integrand in ((7.2)). These results lend some weight to the reliability (i.e. convergence) of the values of  $Z'_N(s)/Z_N(s)$  used in the calculations above.

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<sup>4</sup>For example, by adapting the Sieve of Eratosthenes.

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